

Dynamically assisted Sauter-Schwinger effect in inhomogeneous electric fields

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ABSTRACT: Via the world-line instanton method, we study electron-positron pair creation by a strong (but sub-critical) electric field of the profile $E/\cosh^2(kx)$ superimposed by a weaker pulse $E'/\cosh^2(\omega t)$. If the temporal Keldysh parameter $\gamma_\omega = m\omega/(qE)$ exceeds a threshold value $\gamma_\omega^{\text{crit}}$ which depends on the spatial Keldysh parameter $\gamma_k = mk/(qE)$, we find a drastic enhancement of the pair creation probability – reporting on what we believe to be the first analytic non-perturbative result for the interplay between temporal and spatial field dependences $E(t, x)$ in the Sauter-Schwinger effect. Finally, we speculate whether an analogous effect (drastic enhancement of tunneling probability) could occur in other scenarios such as stimulated nuclear decay, for example.

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1 Introduction

Despite the tremendous progress of quantum field theory as a fundamental description of nature, our understanding of its non-perturbative properties is still disappointingly incomplete. In quantum electrodynamics (QED), for example, a striking non-perturbative phenomenon is the Sauter-Schwinger effect predicting the creation of electron-positron pairs out of the vacuum by a strong electric field [1–4]. In case of a constant (sub-critical) electric field $E < E_{\text{crit}}$, the pair creation probability behaves as ($\hbar = c = 1$)

$$P_{e^+e^-} \sim \exp \left\{ -\pi \frac{m^2}{qE} \right\} = \exp \left\{ -\pi \frac{E_{\text{crit}}}{E} \right\}, \quad (1.1)$$

where $E_{\text{crit}} = m^2/q \approx 1.3 \times 10^{18} \text{ V/m}$ denotes the Schwinger critical field. Unfortunately, the dependence of this non-perturbative phenomenon on the field profile $\mathbf{E}(t, \mathbf{r})$ away from the constant field approximation is still mostly *terra incognita*. There are many results for fields which depend on one coordinate only, such as space x or time t (see, e.g., [5–19]), one of the light-cone coordinates $x_{\pm} = t \pm x$ (see, e.g., [20–26]), or other linear combinations of x and t [27]. In these cases, the underlying (Dirac or Klein-Fock-Gordon) equation simplifies to an ordinary differential equation (allowing for a WKB approach, for example, see also [28–31]).

However, to the best of our knowledge, there are no analytic non-perturbative results for fields $E(t, x)$ which genuinely depend on space x and time t . So far, this case has only been treated numerically via the Wigner formalism (see, e.g., [32–34]) or a direct integration of the Dirac equation (see, e.g., [35–37]). This lack of understanding is not only unsatisfactory from a theoretical point of view. A deeper insight into the impact of space-time dependent fields is also highly desirable in view of experimental efforts with lasers

[38–48], for example, aiming at a verification of this non-perturbative pair-creation effect.

¹ In the following, we venture a first step into this direction and employ the world-line instanton technique (see, e.g., [49–57]) in order to study the superposition of a spatial and a temporal field pulse as an example for a genuinely space-time dependent field.

2 World-line instanton method

Let us start with a brief review of the world-line instanton method, see, e.g., [49–57]. Since the electron spin does not affect the exponent of the pair creation probability [52], we consider the vacuum persistence amplitude of scalar QED

$$\langle 0_{\text{out}} | 0_{\text{in}} \rangle = \iint \mathcal{D}\phi \mathcal{D}\phi^* e^{i \int d^4x (|D_\mu \phi|^2 - m^2 |\phi|^2)}, \quad (2.1)$$

with the covariant derivative $D_\mu = \partial_\mu + iqA_\mu$. After analytic continuation to Euclidean space, this functional path integral can be translated into the world-line representation [49] where $\mathcal{D}\phi \mathcal{D}\phi^*$ is replaced by the sum over all closed loops $x_\mu(s)$ in Euclidean space. Then, via the saddle point method (with the electron mass m playing the role of the large expansion parameter), the pair creation probability can be estimated as

$$P_{e^+e^-} = 1 - |\langle 0_{\text{out}} | 0_{\text{in}} \rangle|^2 \sim e^{-\mathcal{S}}, \quad (2.2)$$

with the world-line instanton action ²

$$\mathcal{S} = ma + iq \int_0^1 ds \dot{x}^\mu A_\mu(x^\nu). \quad (2.3)$$

Here $\dot{x}_\mu = dx_\mu/ds$ denotes the proper-time derivative of a closed $x_\mu(s=0) = x_\mu(s=1)$ world-line loop $x_\mu(s)$ as a solution of the instanton equations

$$m\ddot{x}_\mu = iqF_{\mu\nu}\dot{x}^\nu a \quad (2.4)$$

with $\ddot{x}_\mu = d^2x_\mu/ds^2$ and $\dot{x}_\nu\dot{x}^\nu = a^2 = \text{const.}$

3 Sum of Sauter pulses

Now let us apply the world-line instanton method to a space-time dependent electric field

$$\mathbf{E}(t, x) = \left(\frac{E}{\cosh^2(kx)} + \frac{E'}{\cosh^2(\omega t)} \right) \mathbf{e}_x \quad (3.1)$$

consisting of a strong spatial Sauter [1] pulse $\propto E$ and a weaker temporal Sauter pulse $\propto E'$ where both field strengths are sub-critical $E' \ll E \ll E_{\text{crit}} = m^2/q$. Furthermore,

¹ Apart from the laser laboratories BELLA (Berkley, USA) and VULCAN (Oxford, UK), which are approaching the strong-field (non-linear) QED regime, we would like to mention the European ELI program, the Russian XCELS initiative, or the Chinese SIOM facility, for example.

² Although one could also consider complex instantons, $x^\mu(s)$ is purely real in our case while \hat{A}_μ is purely imaginary, cf. Eq. (3.3).

in order to be in the non-perturbative regime, we assume slowly varying pulses $\omega, k \ll m$. For convenience, we introduce the spatial and temporal Keldysh [58] parameters via

$$\gamma_k = \frac{mk}{qE}, \quad \gamma_\omega = \frac{m\omega}{qE}. \quad (3.2)$$

It will be most convenient to represent the spatial pulse by the scalar potential $A_0(x)$ but the temporal pulse by the vector potential $A_1(t)$. Then, after Wick rotation, the Euclidean vector potential reads

$$A_0(x_1) = i \frac{E}{k} \tanh(kx_1), \quad A_1(x_0) = i \frac{E'}{\omega} \tan(\omega x_0), \quad (3.3)$$

with $x_0 = it$ and $x_1 = x$ as well as $A_2 = A_3 = 0$. As a result, the instanton equations (2.4) assume the form

$$\begin{aligned} \ddot{x}_0 &= + \frac{qEa}{m} \left(\frac{1}{\cosh^2(kx_1)} - \frac{E'}{E} \frac{1}{\cos^2(\omega x_0)} \right) \dot{x}_1, \\ \ddot{x}_1 &= - \frac{qEa}{m} \left(\frac{1}{\cosh^2(kx_1)} - \frac{E'}{E} \frac{1}{\cos^2(\omega x_0)} \right) \dot{x}_0, \end{aligned} \quad (3.4)$$

and are analogous to the planar motion of a charged particle in a magnetic field $\mathbf{B}(\mathbf{r}) = B(x, y)\mathbf{e}_z$.

Due to $E'/E \ll 1$, the second term is negligible unless $\cos^2(\omega x_0)$ becomes very small – which happens near the poles of $E(x_0, x_1)$ at $\omega x_0 = \pm\pi/2$. Away from these poles, we may omit the second term and the above equations can be integrated approximately to

$$\begin{aligned} \dot{x}_0 &= \frac{a}{\gamma_k} \tanh(kx_1) + ab, \\ \dot{x}_1 &= \pm a \sqrt{1 - \left(\frac{\tanh(kx_1)}{\gamma_k} + b \right)^2}. \end{aligned} \quad (3.5)$$

As mentioned after Eq. (2.4), the constant a is given by $\dot{x}_\nu \dot{x}^\nu = a^2 = \text{const}$. The other integration constant b determines the velocity \dot{x}_0 just before (or just after) crossing the x_0 -axis, see Fig 1.

Near the poles $\omega x_0 \approx \pm\pi/2$, on the other hand, the second term becomes important. Similar to the reflection of a charged particle at the region of a very strong magnetic field, the instanton trajectory is basically reflected by the “wall” at $\omega x_0 \approx \pm\pi/2$ if it reaches out far enough. Since this reflection occurs during a very short proper time Δs , we may neglect the regular terms in Eq. (3.4) and keep only the divergent contributions. Then, the equation for x_1 can be integrated approximately to

$$\dot{x}_1 \approx \frac{qE'a}{m\omega} \tan(\omega x_0) + \dot{x}_1^{\text{in}}, \quad (3.6)$$

and thus the equation for x_0 becomes

$$\ddot{x}_0 \approx - \frac{(qE'a)^2}{m^2\omega} \frac{\tan(\omega x_0)}{\cos^2(\omega x_0)} \sim \frac{1}{(\omega x_0 \pm \pi/2)^3}. \quad (3.7)$$

As a result, the perpendicular velocity \dot{x}_0 is reversed by that reflection while the parallel velocity \dot{x}_1 has the same value \dot{x}_1^{in} before and after the reflection, see Fig 1.

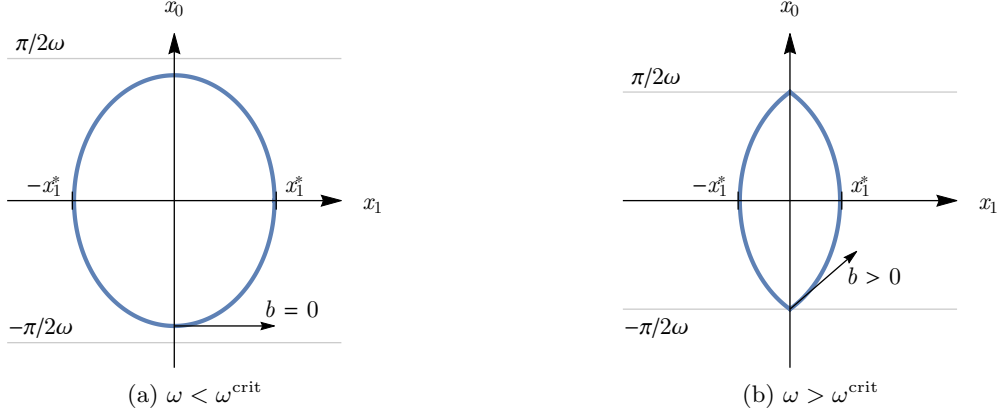


Figure 1: Sketch of the instanton trajectories in the Euclidean x_0, x_1 -plane: Left (a): for frequencies below threshold $\omega < \omega^{\text{crit}} \leftrightarrow \gamma_\omega < \gamma_\omega^{\text{crit}}$ with $\gamma_\omega^{\text{crit}}$ being given by Eq. (4.6), the trajectory does not reach the poles at $\pm\pi/(2\omega)$ and thus it has nearly the same form as in the absence of the temporal pulse $\propto E'$. Right (b): for frequencies above threshold $\omega > \omega^{\text{crit}} \leftrightarrow \gamma_\omega > \gamma_\omega^{\text{crit}}$, the instanton trajectory is effectively reflected at the poles.

4 Tunneling probability

From Eq. (3.7), we may estimate the contribution to the instanton action (2.3) stemming from the short proper-time period of the reflection and obtain that it is suppressed as $(E'/E) \ln(E'/E)$ for small E' . Thus, for $E \gg E'$, we may approximate the instanton action by the part between the reflections

$$\mathcal{S} \approx ma - \frac{qE}{k} \int_0^1 ds \tanh(kx_1) \dot{x}_0. \quad (4.1)$$

In order to calculate the above integral, we split the closed loop in Fig. 1(b) into four quarters: starting at the lower reflection point $x_0 = -\pi/(2\omega)$ with $x_1 = 0$, we go to the spatial turning point x_1^* (where $x_0 = 0$), then from there to the upper reflection point $x_0 = +\pi/(2\omega)$ with $x_1 = 0$, and then to $-x_1^*$ (where $x_0 = 0$), and finally back to $x_1 = 0$ and $x_0 = -\pi/(2\omega)$. Since each quarter yields the same contribution, we get

$$\mathcal{S} \approx ma - \frac{4m}{\gamma_k} \int_0^{x_1^*} dx_1 \frac{\tanh(kx_1) (\tanh(kx_1) + \gamma_k b)}{\sqrt{\gamma_k^2 - (\tanh(kx_1) + \gamma_k b)^2}}, \quad (4.2)$$

where x_1^* denotes the spatial turning point given by

$$\tanh(kx_1^*) + \gamma_k b = \gamma_k, \quad (4.3)$$

i.e., the zero of the square root in the integral in Eq. (4.2) where $dx_1/dx_0 = 0$. The constant a is determined by $\dot{x}_\nu \dot{x}^\nu = a^2$ and $x_\mu(s=0) = x_\mu(s=1)$ which gives

$$a = \frac{4}{\gamma_k} \int_0^{x_1^*} \frac{dx_1}{\sqrt{\gamma_k^2 - (\tanh(kx_1) + \gamma_k b)^2}}. \quad (4.4)$$

The remaining integration constant b depends on the frequency ω . If ω is too small and thus the poles at $\omega x_0 = \pm\pi/2$ are too far away, the instanton trajectory is not reflected at all and thus we have $b = 0$, see Fig 1. In case of reflection, the integration constant b is non-zero and determined by the implicit condition

$$\frac{4m}{\gamma_k} \int_0^{x_1^*} dx_1 \frac{\tanh(kx_1) + \gamma_k b}{\sqrt{\gamma_k^2 - (\tanh(kx_1) + \gamma_k b)^2}} = \frac{\pi}{2\omega}. \quad (4.5)$$

Together with the above equations for x_1^* , a , and b , Eq. (4.2) is the main result of this paper.

The threshold condition $b = 0$ translates into

$$\gamma_\omega = \frac{\pi}{2} \frac{\gamma_k \sqrt{1 - \gamma_k^2}}{\arcsin(\gamma_k)} \stackrel{\text{Def}}{=} \gamma_\omega^{\text{crit}}. \quad (4.6)$$

Again, if the frequency is too low $\gamma_\omega < \gamma_\omega^{\text{crit}}$, the instanton trajectory is basically not affected by the poles at $\omega x_0 = \pm\pi/2$ leading to $b = 0$ and thus the weak temporal pulse $\propto E'$ has negligible impact. In this case $b = 0$, we get $x_1^* = \text{artanh}(\gamma_k)/k$ and all the integrals can be carried out analytically, yielding the same results as for a static Sauter pulse, see, e.g., [52]. If the frequency exceeds this threshold value $\gamma_\omega > \gamma_\omega^{\text{crit}}$, on the other hand, the instanton trajectory is reflected at the poles (i.e., $b > 0$) and thus the instanton action (4.2) is reduced by the weak temporal pulse $\propto E'$, leading to a significant enhancement of the pair creation probability. This enhancement is an example of the dynamically assisted Sauter-Schwinger effect, see [64–69], for an inhomogeneous electric field. In the homogeneous limit $\gamma_k \downarrow 0$, the threshold value (4.6) approaches $\gamma_\omega^{\text{crit}} = \pi/2$ consistent with the results of [64]. For $\gamma_k \uparrow 1$, the threshold $\gamma_\omega^{\text{crit}}$ scales as $\gamma_\omega^{\text{crit}} \approx \sqrt{1 - \gamma_k^2}$, i.e., very small frequencies ω can have a significant impact in this case.

Unfortunately, due to the implicit nature of the condition for b , we cannot provide a closed analytical expression for \mathcal{S} . However, near but above threshold, we can expand the involved quantities and obtain the following approximate formula for the instanton action:

$$\mathcal{S} = \frac{m^2}{qE} \begin{cases} \frac{2\pi}{1 + \sqrt{1 - \gamma_k^2}}, & \text{for } \gamma_\omega \leq \gamma_\omega^{\text{crit}}, \\ \frac{2\pi}{1 + \sqrt{1 - \gamma_k^2}} - \pi \frac{(1 - \gamma_k^2)^{3/2}}{\gamma_k^2 (\gamma_\omega^{\text{crit}})^4} [\gamma_\omega - \gamma_\omega^{\text{crit}}]^2 + \mathcal{O}([\gamma_\omega - \gamma_\omega^{\text{crit}}]^3), & \text{for } \gamma_\omega > \gamma_\omega^{\text{crit}}. \end{cases} \quad (4.7)$$

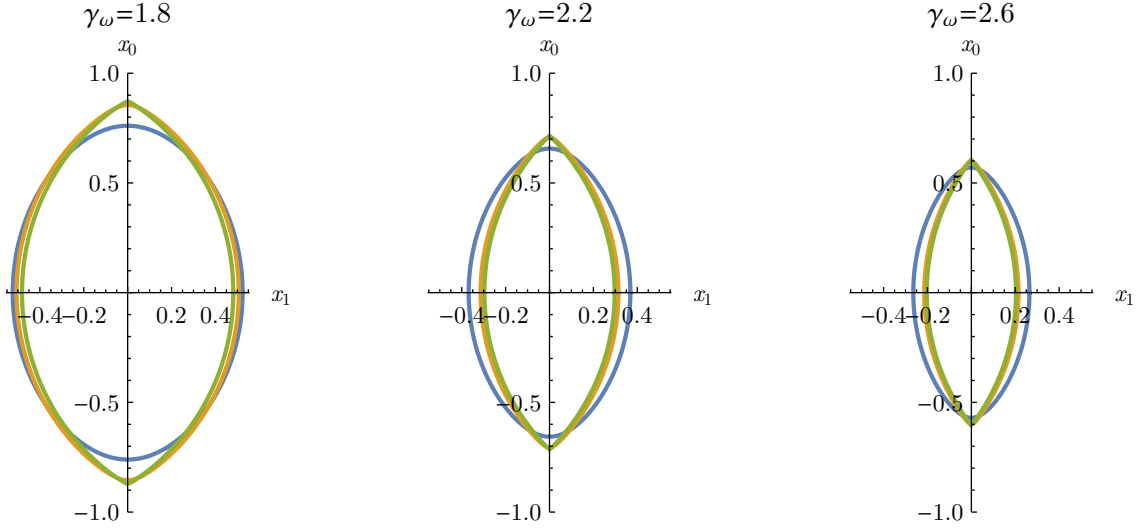


Figure 2: Instanton trajectories [in units of $m/(qE)$] found by numerically integrating (3.4) for different values of γ_ω and $\gamma_k = 0.5$ in all cases. The colors represent $E'/E = 10^{-1}$ (blue), $E'/E = 10^{-2}$ (orange), and $E'/E = 10^{-3}$ (green). The limit $E'/E \rightarrow 0$ used in the analytic approximation (cf. Fig. 1) is indistinguishable from the trajectory with $E'/E = 10^{-3}$.

The first line (valid below and at threshold $\gamma_\omega \leq \gamma_\omega^{\text{crit}}$) is just the result in the static case (see, e.g., [52]). For $\gamma_k = 0$, we recover Schwinger’s result [4] for a constant field (1.1). Above threshold $\gamma_\omega > \gamma_\omega^{\text{crit}}$ on the other hand, the action is reduced by the second-order term $\propto [\gamma_\omega - \gamma_\omega^{\text{crit}}]^2$.

5 Numerical evaluation

Let us now compare the analytic approximation above to a numerical solution of the full instanton equations (3.4). To fulfill the periodicity constraints, a shooting method can be used: we numerically integrate (3.4), varying the initial conditions and a until a closed solution is found using a root finding algorithm. Figure 2 shows these closed solutions for different values of γ_ω and E'/E with a fixed $\gamma_k = 0.5$. As explained above, for small E' the time dependent pulse acts like a reflecting barrier. For finite values of E' such as $E'/E = 10^{-1}$, this barrier is “softened” and the trajectories (blue curves in Fig. 2) differ a bit from the analytical approximation in Fig. 1. For smaller E'/E however, they converge to the analytical approximation (orange and green curves in Fig. 2) and already for $E'/E = 10^{-3}$ (green curves in Fig. 2) they are virtually indistinguishable from the analytical approximation in Fig. 1. They perfectly agree with the reflection picture, further justifying the approximations used above.

With these solutions to the instanton equations, we can evaluate the action (2.3). An exemplary section with $\gamma_k = 0.3$ is shown in Figure 3, left panel. While the results for $E'/E = 10^{-1}$ (blue crosses) deviate a bit from the analytical approximation (red line) mainly because the total effective field is a bit larger than E , the agreement for $E'/E = 10^{-3}$

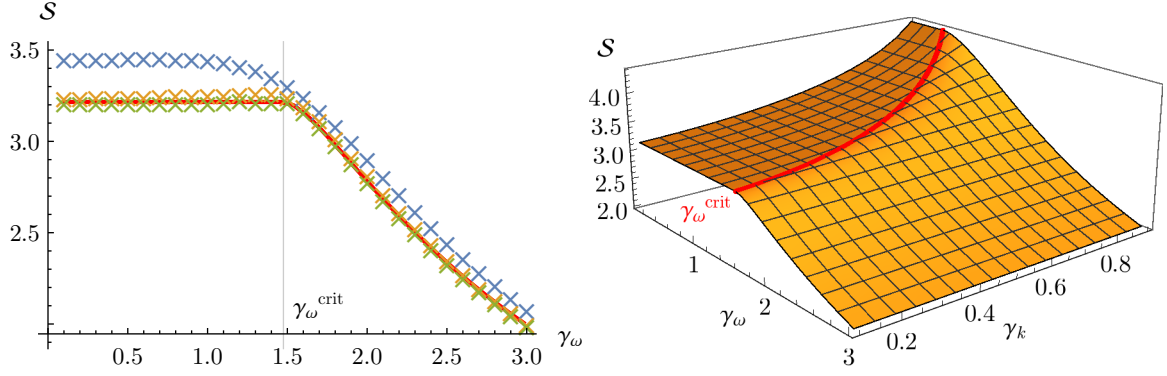


Figure 3: Left: Comparison of the action (2.3) found by numerically integrating (3.4) for $\gamma_k = 0.3$ and $E'/E = 10^{-1}$ (blue), $E'/E = 10^{-2}$ (orange), and $E'/E = 10^{-3}$ (green) to the integral (4.2) (red line). Right: (4.2) evaluated for varying γ_ω and γ_k . The thick, red line visualizes the threshold $\gamma_\omega^{\text{crit}}(\gamma_k)$ in (4.6). The action \mathcal{S} is displayed in units of $m^2/(qE)$.

(green crosses) is excellent. Assembling all these sections for different values of γ_k together, we obtain the landscape plot in Figure 3, right panel. Here we plotted the analytical result, but the landscape obtained numerically for $E'/E = 10^{-3}$, for example, would be indistinguishable. The red curve in Figure 3, right panel, marks the threshold $\gamma_\omega^{\text{crit}}(\gamma_k)$ in (4.6).

6 Conclusions

Via the world-line instanton technique, we derived an analytical estimate (4.2) for the electron-positron pair creation probability (2.2) induced by an electric field (3.1) which genuinely depends both on space and on time. Superimposing a strong spatial pulse by a weak temporal pulse (3.1), we found that the weak pulse is negligible for small frequencies $\gamma_\omega \leq \gamma_\omega^{\text{crit}}$ but can enhance the pair creation probability significantly (dynamically assisted Sauter-Schwinger effect) for larger frequencies $\gamma_\omega > \gamma_\omega^{\text{crit}}$ with the threshold (4.6) depending on the spatial Keldysh parameter (3.2). Note that we treated both, the strong $\propto E$ and the weak $\propto E'$ field non-perturbatively. This becomes evident by the reflection of the instanton trajectory in Fig. 1, for example, which is a drastic change and thus cannot be observed by a perturbative expansion around the strong field solution (see, e.g., [65]). Similarly, the space and time dependence is fully taken into account – the threshold behavior (4.6) requires going beyond the locally constant field approximation or gradient expansion (see, e.g., [69]).

In the homogeneous limit $\gamma_k \downarrow 0$, this threshold $\gamma_\omega^{\text{crit}}$ converges to $\pi/2$ in accordance with [64]. If the spatial Keldysh parameter approaches unity $\gamma_k \uparrow 1$, on the other hand, the threshold $\gamma_\omega^{\text{crit}}$ goes to zero. In this case $\gamma_k \uparrow 1$, the size of the spatial Sauter pulse is barely enough to produce electron-positron pairs and the instanton loop becomes very large, cf. $x_1^* = \text{artanh}(\gamma_k)/k$ for $b = 0$. This equation $x_1^* = \text{artanh}(\gamma_k)/k$ can be rewritten as $m = qA_0(x_1^*)$ which shows that the potential difference between the spatial turning points

$+x_1^*$ and $-x_1^*$ equals the energy gap of $2mc^2$. For $\gamma_k \uparrow 1$, the total asymptotic electrostatic potential difference approaches this energy gap and thus the spatial turning points $\pm x_1^*$ move to infinity. Quite intuitively, even comparably small frequencies (leading to poles at large distances to the origin) can have an impact in this limit.

Note that we focused on the exponent \mathcal{S} in the electron-positron pair creation probability (2.2) given by the instanton action (2.3) in this work. The pre-factor in front of the exponential can also be derived (at least numerically) via the world-line instanton method by studying small perturbations around the instanton trajectory³. Hence, this pre-factor is almost unaffected by the weak pulse $\propto E'$ below threshold $\gamma_\omega < \gamma_\omega^{\text{crit}}$, but it is expected to depend on E' above threshold $\gamma_\omega > \gamma_\omega^{\text{crit}}$. Thus, for extremely small E' , this pre-factor could partly counteract the exponential enhancement of the pair creation probability. However, since the parameter dependence (e.g., power-law) of this pre-factor is sub-dominant compared to the exponential dependence on \mathcal{S} , there will be a large region of parameter space where the dynamically assisted Sauter-Schwinger effect applies⁴.

In order to get a feeling for the orders of magnitude of the involved parameters, let us insert some potentially realistic example values. An optical laser focus with a focal field strength of $E = 10^{16}$ V/m corresponding to an intensity of order 10^{25} W/cm² has a very small γ_k . Hence we get $\gamma_\omega^{\text{crit}} \approx \pi/2$ corresponding to a threshold frequency of $\omega^{\text{crit}} \approx 6$ keV, which is well below the energy gap $2mc^2 \approx 1$ MeV and could be reached by an x-ray free-electron laser (XFEL) or by higher harmonic focusing [70], for example. If the strong field itself is generated by such high-energy sources, on the other hand, the resulting γ_k can approach unity which lowers $\gamma_\omega^{\text{crit}}$ and thus ω^{crit} . An XFEL with 12 keV, for example, has $\gamma_k \approx 1/2$ and thus we get $\omega^{\text{crit}} \approx 5$ keV while an XFEL with 24 keV corresponds to $\gamma_k \approx 1$ such that ω^{crit} vanishes. In these cases, the actual frequency ω exceeds the threshold frequency ω^{crit} indicating that the dynamically assisted Sauter-Schwinger effect plays a role.

However, these numbers should only be understood as an illustration because the profile (3.1) does not portray a real laser pulse. This motivates the study of more realistic field configurations, for example transverse fields such as $A_z(t, x)$. Preliminary investigations indicate that the world-line instanton method can also be applied to these more complex field profiles and that the general behavior of dynamically assisted Sauter-Schwinger effect persists. This will be the subject of future studies, see also [71].

7 Outlook

Going a step beyond the scenario considered here, we may ask the question of what we can learn from this analysis for more general tunneling phenomena in the presence of additional time-dependent fields. One lesson might be that the characteristic time scale

³ Note that, in contrast to the instanton action \mathcal{S} , the pre-factor does depend on the spin, i.e., may differ for scalar and spinor QED.

⁴ If E' becomes too small, the approach used here breaks down eventually. On the one hand, the small value of E' invalidates the large- m expansion (i.e., the instanton method), and, on the other hand, approximating E' by a classical field is no longer justified.

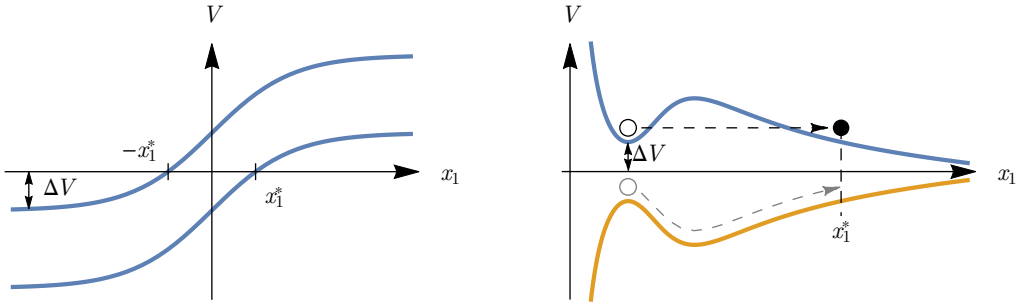


Figure 4: Sketch of the potential landscape $V(x)$ for the spatial Sauter profile $E(x) = E/\cosh^2(kx)$ (left) in comparison to non-relativistic tunneling out of a local minimum (right), as well as the inverted potential $-V(x)$ which is “felt” by the non-relativistic instanton. In both cases, the spatial turning points x_1^* move to infinity if the energy gain ΔV tends to zero.

of the additional temporal dependence required for enhancing the tunneling probability is not set by the height of the tunneling barrier, for example, but by the imaginary turning time x_0^* of the associated instanton solution (in our case $\omega x_0^* = \pm\pi/2$). In non-relativistic quantum mechanics, this imaginary turning time has been identified with the traversal time for tunneling [72], i.e., the time the particle needs for tunneling through the barrier, but this identification is not undisputed.

Another lesson might be the relation between the final kinetic energy of the particle (after tunneling) and the turning points in space and (imaginary) time of the associated instanton solution. For example, let us consider a general smooth and stationary potential landscape $V(x)$ which contains a local minimum and vanishes at spatial infinity, see Fig. 4. In non-relativistic quantum mechanics, the probability for tunneling from the local minimum out to infinity can also be estimated via an instanton solution, which basically corresponds to the motion of the particle within the inverted potential landscape $-V(x)$. Now, if the kinetic energy at infinity (i.e., ΔV) becomes very small – which corresponds to the limit $\gamma_k \uparrow 1$ in the Sauter-Schwinger scenario – the spatial turning point x_1^* and thus also the imaginary turning time x_0^* become very large, suggesting that small ω can have a large impact.

As a potential application, one could envisage nuclear α -decay, for example. In the usual Gamow picture [73], this process can be described by tunneling through a potential barrier which is dominated by the Coulomb repulsion far away from the nucleus while the nuclear attraction prevails at smaller distances. The height of this barrier is typically a few tens of MeV while its width, i.e., the spatial turning point x_1^* , depends on the final kinetic energy of the α -particle E_α .

Because the instanton trajectory grows very large for small E_α/m_α , the asymptotic behavior of the field (for large x_1) dominates since the “instanton spends most of its time there”, see also [74]. We can calculate both the turning points and the instanton action to leading order in E_α/m_α by considering an electric field that asymptotically corresponds to

a Coulomb potential. Identifying $\gamma_k = (1 + E_\alpha/m_\alpha)^{-1}$ the spatial turning point is

$$x_1^* = \frac{q_\alpha q_{\text{nucleus}}}{4\pi\varepsilon_0 m_\alpha} \frac{1}{\gamma_k(1 - \gamma_k)} \approx \frac{q_\alpha q_{\text{nucleus}}}{4\pi\varepsilon_0 E_\alpha}, \quad (7.1)$$

which is quite large (in comparison to other nuclear length scales) for typical decays. For example, ^{238}U has $E_\alpha \approx 4.3$ MeV and $x_1^* \approx 62$ fm. The imaginary turning time is given by an integral similar to (4.5):

$$x_4^* \approx \frac{m_\alpha}{q_\alpha E} \int_0^1 \frac{du}{(1 - \gamma_k \sqrt{1 - u^2})^2} \approx \frac{m_\alpha}{q_\alpha E} \frac{\pi}{(2E_\alpha/m_\alpha)^{3/2}}, \quad (7.2)$$

so the critical threshold for dynamical assistance in this case is

$$\omega^{\text{crit}} = \frac{\pi}{2x_4^*} \approx \frac{\pi}{Z} \frac{\varepsilon_0 m_\alpha}{q^2} \left(2 \frac{E_\alpha}{m_\alpha}\right)^{3/2}. \quad (7.3)$$

Again, for a typical decay, this is on the order of 100 keV (e.g., for ^{238}U , we get $\omega^{\text{crit}} = 157$ keV), far less than the other characteristic energy scales in the problem, such as the α -particle mass, its final kinetic energy or the barrier height $Zq^2/2\pi\varepsilon_0 R_{\text{nucleus}}$ (≈ 35 MeV for ^{238}U) used in the usual tunneling model.

Let us also calculate the instanton action (and thus the lifetime of the decay) analogously to (4.2). We have to slightly adapt the integral expression for the action in this case, because in the Sauter-Schwinger scenario (Fig. 4, left) there are two spatial turning points $\pm x_1^*$ moving to $\pm\infty$ as $\Delta V \rightarrow 0$, whereas for the α -particle (Fig. 4, right) there is only one $x_1^* \rightarrow \infty$. This amounts to a factor of $\frac{1}{2}$ in the leading-order expression for the instanton action:

$$\begin{aligned} \mathcal{S} &\approx \frac{1}{2} \frac{q_\alpha q_{\text{nucleus}}}{4\pi\varepsilon_0} \frac{4}{\gamma_k^2} \int_0^{\frac{1}{1-\gamma_k}} du \sqrt{\gamma_k^2 - \left(1 - \frac{1}{u}\right)^2} \\ &\approx \frac{q_\alpha q_{\text{nucleus}}}{2\varepsilon_0} \frac{1}{\sqrt{2E_\alpha/m_\alpha}} \\ &= \frac{q^2 \sqrt{m_\alpha}}{\varepsilon_0} \frac{Z}{\sqrt{E_\alpha}}. \end{aligned} \quad (7.4)$$

To leading order in E_α/m_α , we reproduce the well known Geiger-Nuttall law [75]

$$\ln(P) \propto -\mathcal{S} \propto -\frac{Z}{\sqrt{E_\alpha}}. \quad (7.5)$$

The expression (7.4) is also obtained as the leading-order contribution in the WKB formula for an α -particle tunneling through a static Coulomb potential. Via the imaginary turning time in (7.2), we may now estimate the possibility of dynamically assisting this tunneling process.

Note that the mechanism sketched above is quite different from enhanced β -decay and similar ideas (see [76–79] for an incomplete selection) where the available phase space for the emitted or converted electrons or positrons is modified by the electromagnetic field.

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